8). max. $z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

S.t.

$$2x_1 + 5x_2 \leq 98$$

$$x_1, x_2 \geq 0$$

(use Kuhn-Tucker condⁿ)

Solution.

Here, $f(X) = 2x_1^2 + 12x_1x_2 - 7x_2^2$

$$h(X) = 2x_1 + 5x_2 - 98 \leq 0$$

where, $X = (x_1, x_2)$.

Hessian matrix is

$$H^B = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 12 \\ 12 & -14 \end{bmatrix}$$

∴ The principal minors are

$\Delta_1 = 4$

$\Delta_2 = -56 - 144 = -200 < 0$

which are of the alternate sign.

Hence, $f(X)$ is a concave funⁿ of x_1 and x_2

Also, $h(X) = 2x_1 + 5x_2 - 98 \leq 0$
is a convex function

Hence Kuhn-Tucker Necessary and conditions for maximum value of $f(X)$ are also the sufficient cond^n.

The necessary cond^n for existence of max^m value of $f(X)$ are

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\Rightarrow 4x_1 + 12x_2 - 2\lambda = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\Rightarrow 12x_1 - 14x_2 - 5\lambda = 0 \quad \text{--- (2)}$$

$$\lambda h(X) = 0 \Rightarrow \lambda(2x_1 + 5x_2 - 98) = 0 \quad \text{--- (3)}$$

$$h(X) \leq 0 \Rightarrow 2x_1 + 5x_2 - 98 \leq 0 \quad \text{--- (4)}$$

$$\lambda \geq 0 \quad \text{--- (5)}$$

5 $\Rightarrow$ either $\lambda = 0$ or $\lambda > 0$

If $\lambda = 0$ then 1 $\Rightarrow 4x_1 + 12x_2 = 0$

2 $\Rightarrow 12x_1 - 14x_2 = 0$

on solving, we get $x_1 = 0, x_2 = 0$
which is the initial value of the fun^n $f(X)$ if $\lambda = 0$

Now, if $\lambda \neq 0$, i.e. $\lambda > 0$

1 $\Rightarrow 2x_1 + 6x_2 - \lambda = 0$ —— 6

2 $\Rightarrow 12x_1 - 14x_2 - 5\lambda = 0$ —— 7

eq^n 6 $\times 6 -$ 7

$$\cancel{12x_1} + 36x_2 - 6\lambda = 0$$
$$\cancel{12x_1} - 14x_2 - 5\lambda = 0$$
$$- \quad + \quad +$$

$$50x_2 - \lambda = 0$$

$$\Rightarrow \boxed{x_2 = \frac{\lambda}{50}}$$

eq^n 6 $\times 7$ + eq^n 7 $\times 3$

$$14x_1 + 42x_2 - 7\lambda = 0$$
$$36x_1 - 42x_2 - 15\lambda = 0$$
$$-\quad\quad + \quad\quad +$$

$$50x_1 - 22\lambda = 0$$

$$\boxed{x_1 = \frac{22}{50}\lambda}$$

since $\lambda \neq 0$

$\therefore$ eq^n 3 $\Rightarrow 2x_1 + 5x_2 - 98 = 0$

$$\Rightarrow 2 \times \frac{22}{50}\lambda + 5 \times \frac{\lambda}{50} = 98$$

$$\Rightarrow \frac{49}{50}\lambda = 98$$

$$\Rightarrow \boxed{\lambda = 100}$$

$$\therefore x_1 = \frac{22}{50} \times 100 = 44$$

$$x_2 = \frac{100}{50} = 2.$$

$$\therefore \max. Z = 2(44)^2 + 12 \times 44 \times 2 - 7(2)^2$$
$$= \underline{4900}$$